Elméleti fizikai módszerek a környezettudományban

5. gyakorlat

- $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3), \mathbf{v} = (v_1(x,y,z), v_2(x,y,z), v_3(x,y,z))$
  $\mathbf{r} = (x, y, z)$ vagy $\mathbf{r} = (x_1, x_2, x_3)$ és $r = \sqrt{x^2 + y^2 + z^2}$
  $\varphi(x, y, z)$ skalár

- vektor-skalár fv.: $\varphi(\mathbf{r}) = \varphi(x, y, z) = xy \sin z + e^{\sqrt{x}y}$
  vektor-vektor fv.: $\mathbf{v}(\mathbf{r}) = \mathbf{v}(x, y, z) = (x^y, \cos(xy), -y \sin z^3) = x^y\mathbf{i} + \cos(xy)\mathbf{j} + (-y \sin z^3)\mathbf{k}$

- skalárszorzat: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i$
  vektoriális szorzat:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

- differenciáloperátorok: $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}) = (\partial_1, \partial_2, \partial_3)$

  gradiens: $\mathrm{grad}\varphi = \nabla\varphi = (\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}) = (\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}) = (\partial_1\varphi, \partial_2\varphi, \partial_3\varphi)$
  $(\mathrm{grad}\varphi)_i = \partial_i\varphi$

  divergencia: $\mathrm{div}\mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \sum \frac{\partial v_i}{\partial x_i} = \sum \partial_i v_i$
  $\mathrm{div}\mathbf{v} = \sum \partial_i v_i$

  rotáció: $\mathrm{rot}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{pmatrix}$

  $(\mathrm{rot}\mathbf{v})_i = \sum_{j,k} \varepsilon_{ijk} \partial_j v_k$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{, ha } ijk = 123, 231, 312 \\ -1 & \text{, ha } ijk = 132, 213, 321 \\ 0 & \text{, ha legalább 2 megegyezik közülük} \end{cases}$$

  Laplace-operátor: $\Delta \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \sum \partial_i^2$

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1. $\mathbf{v} = \left(2^{-x+y}, 8^{-y+z}, 4^{x-z}\right)$, $\mathrm{div}\mathbf{v} = ?$, $\mathrm{rot}\mathbf{v} = ?$
2. $\mathrm{grad} r^3 = ?$
3. $\mathrm{rot}\,\mathrm{grad} r^3 = ?$
4. $\mathrm{div}\,\mathrm{grad} r^3 = ?$
5. $\mathbf{v} = \left(-\frac{y}{r^3}, \frac{x}{r^3}, 0\right)$, $\mathrm{div}\mathbf{v} = ?$
6. $\mathbf{v} = (cx, cy, 0)$, $\mathrm{div}\mathbf{v} = ?$, $\mathrm{rot}\mathbf{v} = ?$
7. $\mathbf{v} = \left(c(x^2 + y^2), c(x^2 + y^2), 0\right)$, $\mathrm{div}\mathbf{v} = ?$, $\mathrm{rot}\mathbf{v} = ?$
8. $\mathbf{v} = \left(x^2 y z^3, y^2 z, x^2 z\right)$, $\mathrm{div}\mathbf{v} = ?$, $\mathrm{rot}\mathbf{v} = ?$
9. $\mathbf{v} = \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)$, $\mathrm{div}\mathbf{v} = ?$
10. $\mathbf{v} = \left(xr^3, yr^3, zr^3\right)$, $\mathrm{div}, \mathbf{v} = ?$, $\mathrm{rot}\mathbf{v} = ?$

11. $\mathbf{v} = c\dfrac{\mathbf{r}}{r^2}$, $\mathrm{div}\mathbf{v} = ?$, $\mathrm{rot}\mathbf{v} = ?$

12. $\mathbf{v} = \left(r, r^2, r^3\right)$, $\mathrm{div}\mathbf{v} = ?$, $\mathrm{rot}\mathbf{v} = ?$

13. $\mathbf{v} = \mathbf{r}e^r$, $\mathrm{div}\mathbf{v} = ?$, $\mathrm{rot}\mathbf{v} = ?$